

An interquark potential model for multi-quark systems

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Abstract

A potential model for four interacting quarks is constructed in $SU(2)$ from six basis states – the three partitions into quark pairs, where the gluon field is either in its ground state or first excited state. With four independent parameters to describe the interactions connecting these basis states, it is possible to fit 100 pieces of data – the ground and first excited states of configurations from six different four-quark geometries calculated on a $16^3 \times 32$ lattice.

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I. INTRODUCTION

Over the years there has been much progress in lattice QCD. However, this has been restricted to systems with only a few quarks – upto three in most cases. On the other hand, in particle and nuclear physics there is also considerable interest in few- and multi-*hadron* systems beginning with the possibility of bound $K\bar{K}$ states. It seems unlikely that direct results for any but the simplest of these hadron systems could be obtained in the near future from lattice QCD simulations. Our approach is thus to construct a model that reproduces lattice results for the simplest multi-quark systems in a way that can be relatively easily extended to more complex cases. The lattice data has been obtained earlier by the Helsinki group [1](and references therein) as energies of four static quarks in various geometries, such as quarks at the corners of squares, rectangles, tetrahedra and some other less symmetric geometries. This 'experimental' data is then to be explained by a model. The choice of geometries is hopefully general enough for the model to reproduce also the energies for geometries intermediate to those actually considered.

It should be pointed out that the 'experimental' data is not for full QCD. For numerical reasons, the SU(2) gauge group is used instead of SU(3), the use of the latter requiring an order of magnitude increase in computer resources. However, the indications are that, for the present type of discussion, SU(2) suffices. Another approximation is taking the quarks to have an infinite mass. This means that our constituent quarks are static and that there are no sea quarks (the so-called quenched approximation). At present, this limit is now being partially removed by applying the techniques of Ref. [2] to a system of two light and two heavy (static) quarks. The sea quarks will appear through using gauge configurations generated by the UKQCD collaboration for full QCD. However, in this paper, we only attempt to understand systems with four static SU(2) quarks in the quenched approximation.

Any model that can be extended to multi-quark systems must presumably depend explicitly only on the quark degrees of freedom, with the gluon degrees of freedom entering only implicitly. This is the same philosophy that has been successful for interacting multi-nucleon systems. In the latter, the meson fields generating the interactions are 'summarized' as internucleon potentials, which take the form of two-nucleon potentials and, to a lesser extent, three- and four-nucleon potentials. However, an important difference between multi-quark and multi-nucleon systems is the strongly self-interacting nature of the gluon fields mediating the interaction in the former case. It is, therefore, not at all clear that *any* effective model defined in terms of only quark degrees of freedom will be successful. It is the purpose of this article to see to what extent a model can be developed for the four-quark case. If this attempt fails, then there is no point in expecting it to work in even more complex quark problems; a successful model for four quarks is necessary but not sufficient before considering any extension to even more quarks.

It should be added that, in spite of the problems outlined above, many groups still consider that multi-quark systems can be treated using simply the basic two-quark potential e.g. [3,4]. In the opinion of the authors – and backed up by their earlier works – there seems to be no justification for such an approach.

In Section 2 three versions of the model, starting with the simplest, are introduced and in Section 3 results are given.

II. THE MODEL WITH 2, 3 OR 6 BASIS STATES

Since the four quarks in the lattice calculation are static, the corresponding model should not contain any kinetic energy. Also, because we only consider SU(2), there is no distinction between the group properties of quarks(q) and antiquarks(\bar{q}). Four such quarks can then be partitioned as pairs in three different ways

$$A = (q_1 q_3)(q_2 q_4), \quad B = (q_1 q_4)(q_2 q_3) \quad \text{and} \quad C = (q_1 q_2)(q_3 q_4), \quad (2.1)$$

where each $(q_i q_j)$ is a colour singlet. However, these three basis states are not orthogonal to each other. Also, remembering the fact that the quarks are indeed fermions gives, in the weak coupling limit, the condition in the appendix of Ref. [5]

$$|A + B + C\rangle = 0. \quad (2.2)$$

Since $\langle A|A\rangle = \langle B|B\rangle = \langle C|C\rangle = 1$, we get – in this limit – the equalities $\langle A|B\rangle = \langle B|C\rangle = \langle A|C\rangle = -1/2$.

A. 2 basis states

Restricting the basis to only the two states A and B leads to the normalisation matrix

$$\mathbf{N} = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}. \quad (2.3)$$

In addition to this there is the potential matrix

$$\mathbf{V} = \begin{pmatrix} v_{AA} & V_{AB} \\ V_{BA} & v_{BB} \end{pmatrix}. \quad (2.4)$$

In the weak coupling limit – when all quarks are close to each other – we expect the potential matrix to simply become that form appropriate for one-gluon exchange i.e.

$$V_{ij} = -\frac{1}{3} \sum_{i \leq j} \vec{\tau}_i \cdot \vec{\tau}_j v_{ij}, \quad (2.5)$$

where $v_{ij} = -e/r_{ij}$. Then $V_{AA} = v_{13} + v_{24}$ etc. and

$$V_{AB} = V_{BA} = -\frac{1}{2} (v_{13} + v_{24} + v_{14} + v_{23} - v_{12} - v_{34}). \quad (2.6)$$

The object is then to compare the eigenvalues of

$$[\mathbf{V} - E(4)\mathbf{N}] = 0 \quad (2.7)$$

with the lattice results – the success or failure of the model being to what extent the two agree. Here $E(4)$ is the total four-quark energy. However, in practice, it is more convenient to deal with the four-quark *binding* energy E defined as $E = E(4) - V_{AA}$, assuming that the A partition has the lowest energy. A perturbative calculation to $O(\alpha^2)$ has reproduced this two-state model [6].

If all three basis states are included, the model has a problem since the matrix in Eq. 2.7 is singular for the obvious reason that $|A + B + C\rangle = 0$. In some of our earlier work this was interpreted to mean that it was unnecessary to include all three states and so the symmetry was broken by keeping the two states with the lowest energy, let us say, A and B . A similar thing also occurred in the lattice simulations. There it was found that the energy of the lowest state was always the same in both a 2×2 and 3×3 description, providing A or B had the lowest energy. In addition the energy of the second state was, in most cases, more or less the same – the largest difference occurring with the tetrahedral geometry. Another modification to the basic model in Eqs. 2.5–2.7 is to replace the original one-gluon-exchange potential by the full two-quark potential

$$v_{ij} = -e/r_{ij} + b_s r_{ij} + c. \quad (2.8)$$

This is now a move away from the extreme weak coupling limit. In principle, everything is now known, since the parameters e, b_s, c in v_{ij} can be determined from the same lattice calculation as the four-quark energies. However, this model fails badly, since it still contains the van der Waals long range forces. The main feature of this failure is that the lowest eigenvalue (E_0) is too low and the other (E_1) too high. This suggests that the off-diagonal potential V_{AB} is too large. However, as we go away from the weak coupling limit, it is expected that $\langle A|B \rangle$ will decrease until eventually, in the extreme strong coupling limit, it would vanish. This, therefore, suggests that for some realistic intermediate coupling the off-diagonal normalisation matrix element N_{AB} should also be reduced. This effect we simulate by introducing a factor f , which decreases as the distance between the quarks increases i.e.

$$\langle A|B \rangle = -f/2. \quad (2.9)$$

For internal consistency, this same factor must also be introduced into V_{AB} i.e.

$$V_{AB} \rightarrow fV_{AB} \quad (2.10)$$

otherwise the eigenvalues would depend on the self-energy term c in Eq. 2.8. In Ref. [1] this model had the good feature that, when fitting the data (E_0, E_1) for a given square, only a single f was necessary to fit both energies. Of course, f was dependent on the size of the square, but a reasonable parametrization was

$$f(Ia) = \exp(-b_s k_f S), \quad (2.11)$$

where b_s was the string energy of Eq. 2.8, S the area of the square and $k_f \approx 0.5$. The form of this parametrization was motivated by strong coupling ideas [7–9]. The original hope was that, with k_f determined from the squares, the model should automatically also fit other geometries with S being the appropriate area contained by the four quarks. When the four quarks lie in a plane, the definition of S is clear. However, in non-planar cases the situation is more complicated and so here the area is simply taken to be the average of the sum of the four triangular areas defined by the positions of the four quarks i.e. the faces of the tetrahedon. For example, in the notation of Eq. 2.1, the appropriate area $S(AB)$ for f is

$$S(AB) = 0.5[S(431) + S(432) + S(123) + S(124)], \quad (2.12)$$

where $S(ijk)$ is the area of the triangle with corners at i, j and k . For planar geometries this simply reduces to the expected area. But for non-planar cases this is only an approximation to $S(AB)$ – a more correct area being one that is not necessarily a combination of planar areas but of curved surfaces with minimum areas. These possibilities are discussed in Ref. [10]. It would be feasible to incorporate this refinement here, since only a few (≈ 50) such areas are needed. But for a general situation, in which the positions of the quarks are integrated over, it would become impractical, since the expression for the minimum area itself involves a double integration. In contrast, the area used here is an algebraic expression and is, therefore, more readily evaluated for any geometry. The above model will be referred to as Version Ia.

This model has only one free parameter k_f in f . Another possibility with additional parameters f_0, k_P is

$$f(Ib) = f_0 \exp(-b_s k_f S + \sqrt{b_s} k_P P) \quad (\text{Version Ib}), \quad (2.13)$$

where P is the perimeter bounding S . However, as shown in Ref. [11], this reduces in the continuum limit to the same as Version Ia – the differences at $\beta = 2.4$ being mainly due to lattice artefacts.

Unfortunately, both of these models have the feature that, for regular tetrahedra, they are unable to reproduce a degenerate ground state with a *non-zero* energy, since the two eigenvalues are

$$E_0 = -\frac{f/2}{1 + f/2}[V_{CC} - V_{AA}] \quad \text{and} \quad E_1 = \frac{f/2}{1 - f/2}[V_{CC} - V_{AA}] \quad (2.14)$$

and for regular tetrahedra $V_{CC} = V_{AA}$.

B. 3 basis states

In the previous subsection only two of the possible three basis states A, B, C were used, the original reason for this being the condition in Eq. 2.2. However, once the factor f is introduced as in Eqs. 2.9, 2.10 this condition no longer holds, so that all three states can be incorporated. This apparently leads to the need for two more factors f', f'' defined by

$$\langle A|C \rangle = -f'/2 \quad \text{and} \quad \langle B|C \rangle = -f''/2. \quad (2.15)$$

However, with the parametrization of f as in Eq. 2.11 or 2.13 and the definition of S as in Eq. 2.12, it is seen that $f' = f'' = f$, since S is simply proportional to the area of the faces of the tetrahedron defined by the four quark positions and is *independent* of the state combination used. Therefore, a 3×3 model has the form where the N and V matrices are:

$$\mathbf{N}(f) = \begin{pmatrix} 1 & -f/2 & -f/2 \\ -f/2 & 1 & -f/2 \\ -f/2 & -f/2 & 1 \end{pmatrix} \quad \text{and} \quad (2.16)$$

$$\mathbf{V}(f) = \begin{pmatrix} v_{AA} & fV_{AB} & fV_{AC} \\ fV_{BA} & v_{BB} & fV_{BC} \\ fV_{CA} & fV_{CB} & v_{CC} \end{pmatrix}. \quad (2.17)$$

Unfortunately, this model(II) also has some unpleasant features:

- i) Again for regular tetrahedra all three eigenvalues are zero.
- ii) For a linear geometry, since the 'appropriate' area as defined by Eq. 2.12 vanishes, we get $f = 1$ i.e. we are back to the weak coupling limit and a singular matrix.
- iii) For squares the model gives $E_1 = -E_0$, whereas the predictions of Model I in Eqs. 2.14 seem to be nearer the data.

The regular tetrahedron result is clear. There is only one energy scale in the model, since all the v_{ij} are the same. Therefore, there can not be any excitations. The positive feature of this model is that all three states are treated equally - a point that seems to be necessary for regular tetrahedra.

C. 6 basis states

The above models both have trouble in describing regular tetrahedra. This might be considered a minor point to worry about, since such a configuration is very special. However, our philosophy is that, if *any* configuration cannot be fitted, then the model fails, since then there is no reason to expect configurations not checked explicitly to be fitted. An interesting feature of the regular tetrahedron data is that the lowest state becomes *more* bound as the tetrahedron increases in size with the magnitude of E_0 increasing from $-0.0202(8)$ to $-0.028(3)$ as the d^3 cube containing the tetrahedron increases from $d = 2$ to $d = 4$. This is opposite to what happens with squares, where the magnitude of E_0 decreases from $-0.0572(4)$ to $-0.047(3)$ as d increases from 2 to 5. This indicates that there is coupling to some higher state(s) that becomes more effective as the size increases and suggests that these higher states contain gluon excitation with respect to the A, B, C configurations. Model III, introduced in Ref. [11], therefore, extends Model II by adding three more states A^*, B^*, C^* , where in analogy with Eq. 2.1

$$A^* = (q_1 q_3)_{E_u} (q_2 q_4)_{E_u} \text{ etc.} \quad (2.18)$$

Here $(q_1 q_3)_{E_u}$ denotes a state where the gluon field is excited to the lowest state with the symmetry of the E_u representation of the lattice symmetry group D_{4h} . Because it is an odd parity excitation, A^*, B^*, C^* must contain two such states in order to have the same parity as A, B, C . The excitation energy of an E_u state over its ground state (A_{1g}) counterpart is $\approx \pi/R$ for two quarks a distance R apart. As R increases this excitation energy decreases making the effect of the A^*, B^*, C^* states more important, leading to the effect mentioned above. Here we have assumed that these states arise from a combination of excited states with E_u . However, it is possible that they involve other excitations, e.g.

$$A^* = (q_1 q_3)_{A'_{1g}} (q_2 q_4) \text{ etc.}, \quad (2.19)$$

where the A'_{1g} state is a gluonic excitation with the same quantum numbers as the ground state (A_{1g}). For this case the following formalism would be essentially the same. Another possibility, which is not considered here, is that the relevant excitations are flux configurations where all four quarks, instead of two, are involved in forming a colour singlet. In the strong coupling approximation such states would reduce to two-body singlets due to Casimir scaling of the string tensions - the string tension for a higher representation would be more than double the value of the fundamental string tension, thus preventing junctions of two strings in the fundamental and one in the higher representation. This would happen both in SU(2) and SU(3), the only exception being the unexcited C state in SU(3), which would involve an antitriplet string.

There are now several new matrix elements that need to be discussed for $\mathbf{N}(\mathbf{f})$ and $\mathbf{V}(\mathbf{f})$:

- a) With the addition of the A^*, B^*, C^* states and the antisymmetry condition $A^* + B^* + C^* = 0$ analogous to Eq. 2.1, there are now two more functions $f^{a,c}$ defined as

$$\langle A^* | B^* \rangle = \langle A^* | C^* \rangle = \langle B^* | C^* \rangle = -f^c/2 \text{ and}$$

$$\langle A^*|B \rangle = \langle A^*|C \rangle = \dots \text{ etc. } \dots = -f^a/2. \quad (2.20)$$

Here it is assumed that $f^{a,c}$ are both dependent on S as defined in Eq. 2.12. Since f^c involves only the excited states, it is reasonable to expect it has a form similar to f in Eq. 2.11 i.e.

$$f^c = \exp(-b_s k_c S). \quad (2.21)$$

b) By orthogonality $\langle A|A^* \rangle = \langle B|B^* \rangle = \langle C|C^* \rangle = 0$

c) In the weak coupling limit, from the $A^* + B^* + C^* = 0$ condition, we expect $\langle A|B^* \rangle = \langle B|C^* \rangle = \dots = 0$ at small distances. To take this into account we parametrize f^a as

$$f^a = (f_1^a + b_s f_2^a S) \exp(-b_s k_a S). \quad (2.22)$$

A partial justification of this is given in the Appendix and in Fig 6 of Ref. [11]. In the following fit all three parameters f_1^a, f_2^a, k_a are varied. However, it is found that f_1^a is always consistent with zero – as expected from the above condition $\langle A|B^* \rangle = \dots = 0$. Therefore, in most of this work f_1^a is fixed at zero.

d) For the potential matrix $\mathbf{V}(\mathbf{f})$ the diagonal matrix elements, after the lowest energy V_{DD} amongst the basis states is removed, are

$$\langle A^*|V - V_{DD}|A^* \rangle = v^*(13) + v^*(24) - V_{DD}, \text{ etc.},$$

where $v^*(ij)$ is the potential of the E_u state – a quantity also measured on the lattice along with the four-quark energies and

$$V_{DD} = \min[V_{AA} = v(13) + v(24), V_{BB} = v(14) + v(23), V_{CC} = v(12) + v(34)].$$

However, to allow more freedom in the following fits, we introduce a parameter b_0 , where

$$\langle A^*|V - V_{DD}|A^* \rangle = V_{AA} - V_{DD} + b_0 V_{AA}^*, \quad \langle B^*|V - V_{DD}|B^* \rangle = V_{BB} - V_{DD} + b_0 V_{BB}^*,$$

$$\langle C^*|V - V_{DD}|C^* \rangle = V_{CC} - V_{DD} + b_0 V_{CC}^*, \quad (2.23)$$

where

$$V_{AA}^* = v^*(13) + v^*(24) - V_{AA}$$

$$V_{BB}^* = v^*(14) + v^*(23) - V_{BB}$$

$$V_{CC}^* = v^*(12) + v^*(34) - V_{CC}.$$

In the following fit b_0 is expected to be of the order unity and, indeed, fixing $b_0 = 1$ is found to be a good assumption. The two-quark potentials $v(ij)$ are taken to be more elaborate than the three term form of Eq. 2.8. They are fitted to the lattice data using [12]

$$v(r_{ij}) = 0.562 + 0.0696 r_{ij} - \frac{0.255}{r_{ij}} - \frac{0.045}{r_{ij}^2}. \quad (2.24)$$

Similarly, the excitation with E_u symmetry is fitted by

$$\Delta v = v^*(ij) - v(ij) = \frac{\pi}{r_{ij}} - \frac{4.24}{r_{ij}^2} + \frac{3.983}{r_{ij}^4}. \quad (2.25)$$

The extra terms containing r_{ij}^{-2} and r_{ij}^{-4} are purely for numerical reasons and ensure that the fitted values of $v(ij)$ and $v^*(ij)$ are, on average, well within 1% of the lattice values for all $r_{ij} \geq 2$.

e) There are two types of off-diagonal element

$$\langle A^*|V|B^* \rangle \quad \text{and} \quad \langle A|V|B^* \rangle. \quad (2.26)$$

Unlike V_{AB} in Eq. 2.6, there is now no guide from the one-gluon exchange limit of Eq. 2.5. Therefore, a further assumption is necessary for the form of these matrix elements. Several different forms were tried, but the most successful seems to be the one where the matrix elements needed in $[\mathbf{V} - (E + V_{AA})\mathbf{N}] = 0$ are expressed as follows

$$\begin{aligned} \langle A^*|V + \frac{f^c}{2}V_{DD}|B^* \rangle &= -\frac{f^c}{2} [V_{ABCD} + c_0 V_{ABC}^*] = \langle B^*|V + \frac{f^c}{2}V_{DD}|A^* \rangle \\ \langle A^*|V + \frac{f^c}{2}V_{DD}|C^* \rangle &= -\frac{f^c}{2} [V_{CABD} + c_0 V_{CAB}^*] = \langle C^*|V + \frac{f^c}{2}V_{DD}|A^* \rangle \\ \langle B^*|V + \frac{f^c}{2}V_{DD}|C^* \rangle &= -\frac{f^c}{2} [V_{BCAD} + c_0 V_{BCA}^*] = \langle C^*|V + \frac{f^c}{2}V_{DD}|B^* \rangle, \end{aligned} \quad (2.27)$$

where

$$V_{ABCD} = V_{AA} + V_{BB} - V_{CC} - V_{DD}, \quad V_{CABD} = V_{CC} + V_{AA} - V_{BB} - V_{DD},$$

$$V_{BCAD} = V_{BB} + V_{CC} - V_{AA} - V_{DD},$$

$$V_{ABC}^* = V_{AA}^* + V_{BB}^* - V_{CC}^*,$$

$$V_{CAB}^* = V_{CC}^* + V_{AA}^* - V_{BB}^*, \quad V_{BCA}^* = V_{BB}^* + V_{CC}^* - V_{AA}^*.$$

These are seen to be simply the generalisation of Eq. 2.6 to the interaction involving excited gluon states. Also in analogy with b_0 in Eqs. 2.24, c_0 is a free parameter – hopefully of order unity.

Likewise,

$$\langle A|V + \frac{f^a}{2}V_{DD}|B^* \rangle = -\frac{f^a}{2} \left[V_{ABCD} + a_0 \frac{V_{ABC}^*}{2} \right] = \langle A^*|V + \frac{f^a}{2}V_{DD}|B \rangle \quad (2.28)$$

where, again in analogy with b_0 in Eqs. 2.24, a_0 is a free parameter – hopefully of order unity. However, it should be added that this hope is even less well founded than the one for c_0 .

In the special case of regular tetrahedra $V_{DD} = V_{AA} = V_{BB} = V_{CC}$ and \mathbf{V} reduces to the form

$$\mathbf{V} = \left[\begin{array}{ccc|ccc} V_{AA} & -fV_{AA}/2 & -fV_{AA}/2 & 0 & -f^aV_a/2 & -f^aV_a/2 \\ -fV_{AA}/2 & V_{AA} & -fV_{AA}/2 & -f^aV_a/2 & 0 & -f^aV_a/2 \\ -fV_{AA}/2 & -fV_{AA}/2 & V_{AA} & -f^aV_a/2 & -f^aV_a/2 & 0 \\ \hline 0 & -f^aV_a/2 & -f^aV_a/2 & V_b & -f^cV_c/2 & -f^cV_c/2 \\ -f^aV_a/2 & 0 & -f^aV_a/2 & -f^cV_c/2 & V_b & -f^cV_c/2 \\ -f^aV_a/2 & -f^aV_a/2 & 0 & -f^cV_c/2 & -f^cV_c/2 & V_b \end{array} \right], \quad (2.29)$$

where $V_a = V_{AA} + a_0 V_{AA}^*/2$, $V_b = V_{AA} + b_0 V_{AA}^*$, $V_c = V_{AA} + c_0 V_{AA}^*$. As with all geometries

$$\mathbf{N} = \left[\begin{array}{ccc|ccc} 1 & -f/2 & -f/2 & 0 & -f^a/2 & -f^a/2 \\ -f/2 & 1 & -f/2 & -f^a/2 & 0 & -f^a/2 \\ -f/2 & -f/2 & 1 & -f^a/2 & -f^a/2 & 0 \\ \hline 0 & -f^a/2 & -f^a/2 & 1 & -f^c/2 & -f^c/2 \\ -f^a/2 & 0 & -f^a/2 & -f^c/2 & 1 & -f^c/2 \\ -f^a/2 & -f^a/2 & 0 & -f^c/2 & -f^c/2 & 1 \end{array} \right]. \quad (2.30)$$

The full 6×6 matrix $[\mathbf{V} - (E + V_{AA})\mathbf{N}]$ now factorizes into three 2×2 matrices, two of which are identical – giving the observed degeneracy. These have the form

$$[\mathbf{V} - (E + V_{AA})\mathbf{N}] = \begin{bmatrix} -E(1 + f/2) & -f^a(E - V_a)/2 \\ -f^a(E - V_a)/2 & -E(1 + f^c/2) + V_b + f^c V_c/2 \end{bmatrix} = 0, \quad (2.31)$$

whereas the third 2×2 matrix is

$$[\mathbf{V} - (E + V_{AA})\mathbf{N}] = \begin{bmatrix} -E(1 - f) & f^a(E - V_a) \\ f^a(E - V_a) & -E(1 - f^c) + V_b - f^c V_c \end{bmatrix} = 0. \quad (2.32)$$

In this case the problem reduces to solving two quadratic equations for E . However, away from the regular tetrahedron the complete 6×6 matrix needs to be treated directly.

III. RESULTS

This section is in two parts. In subsection III A the results for the 6 basis state model at $\beta = 2.4$ are given. There it is shown that only 5 of the possible 8 parameters have any significant influence. Also it is pointed out that the restriction to the 2 basis state version of subsection II A is distinctly inferior and that the further restriction to only two-body interactions (i.e. $k_f = 0$ in Eq. 2.11) is for most geometries very poor. In subsection III B, it is shown that the parameters of the model do not change significantly as β increases i.e. as the continuum is approached. Therefore, the parameters extracted at $\beta = 2.4$ could be used directly in, for example, a Resonating Group calculation of a four-quark model of meson-meson scattering as in Ref. [4].

A. The 6 basis state model at $\beta = 2.4$

In Ref. [1] four quark energies have been extracted for a variety of geometries using a $16^3 \times 32$ lattice with $\beta = 2.4$. From these energies, only one hundred – distributed over six geometries – are selected for fitting. Configurations containing flux links of less than two lattice units were not included, since they have strong lattice artefacts. In detail, we use 15 Tetrahedra (T), 6 Squares (S), 12 Rectangles (R), 4 Quadrilaterals (Q), 9 Non-Planar (NP) and 4 Linear (L). Only the lowest two energies ($E_{0,1}$) from the lattice simulation are used. But, as mentioned earlier before Eq. 2.8, the values of these two energies are more or less the same from the 2×2 and 3×3 simulations. In the latter case a third energy (E_2) is in fact also available. However, it is not expected to be very reliable due to the higher excitations it contains. Its main purpose was to improve the estimate on $E_{0,1}$ by reducing the contamination from the higher states. The stability of ($E_{0,1}$) is understandable, since they are the lowest states of given symmetry. For example, with squares their wavefunctions are simply $\psi(E_0) = (A + B)/\sqrt{2}$ and $\psi(E_1) = (A - B)/\sqrt{2}$.

Before commencing a fit the size of the errors on the above data must be decided. The lattice simulations, through the boot-strap method, do indeed produce errors – statistical ones. However, some estimate must also be added for systematic errors. How this is done is somewhat subjective. Here the prescription is to assume all errors must be at least 0.005 and, also, at least 10% of the eigenvalue itself. The former corresponds to about 10% for the largest values of E_0 and amounts to about 8 MeV.

The above 100 pieces of data are fitted with **Minuit** – the Migrad option. In the model there are, in principle, eight parameters: k_f in Eq. 2.11, f_1^a, f_2^a, k_a in Eq. 2.22, k_c in Eq. 2.21, b_0 in Eqs. 2.23 and a_0, c_0 in Eqs. 2.28. The strategy for finding the optimal values of these parameters is as follows. Firstly, f_1^a is fixed at zero and b_0 fixed at unity, as discussed after Eqs. 2.22 and 2.23. Furthermore, preliminary minimizations indicate that $k_c = 0$ is a good approximation and that a_0, f_2^a are strongly correlated. This latter point is understandable, since for regular tetrahedra, where the 6 basis model is needed, the V_{ABCD}, V_{CABD} etc. are zero, so that

$$\langle A|V + \frac{f^a}{2}V_{DD}|B^* \rangle = -b_s f_2^a S \exp(-b_s k_a S) \frac{a_0 V_{ABC}^*}{4}$$

i.e. only the combination $a_0 f_2^a$ arises. To avoid this a_0 is, at first, fixed at four, the value that yields essentially the smallest χ^2/dof . This results in the parameter values shown in the first row of Table I. Releasing k_c and b_0 to be free parameters gives the results in the second row. Finally, releasing also a_0 gives the third row.

Table I shows the following points:

- a) The value of k_c is consistent with zero. This appears to be a common feature with all fits. This suggests that $f^c \approx 1$, i.e. the excited configurations interact amongst themselves simply through 2-body potentials without the 4-quark effect introduced when $k_c \neq 0$.
- b) The second row shows that the expected value of $b_0 = 1.0$ persists even when b_0 is allowed to vary.
- c) The third row shows that there is an instability between f_2^a and a_0 , with both having errors that are $\approx 25\%$ of the parameter value. This instability also leads to a value of b_0 that is much less than the expected value of unity.
- d) In the first two rows $k_0 \approx 3f_2^a$ – in qualitative agreement with the approximate relationship $k_0 = 4f_2^a$ derived in the Appendix.
- e) The fourth row comes from Ref. [13], which contained a preliminary fit to the above data, but with a model in which the off-diagonal matrix elements of Eqs. 2.27, 2.28 had a slightly different form. The outcome, however, is qualitatively rather similar to the present results.

In order to see from where the χ^2 originates, Table II shows the contributions of the average

$$\chi_A^2 = \frac{1}{N(G)} \sum_{i=1}^{N(G)} (E_i - M_i)^2 / \Delta E_i$$

from each of the 12 types of data – i.e. from E_0, E_1 for the six geometries (T, S, R, Q, NP, L) . Here $N(G)$ is the number of data points for geometry G , M_i are the model results and $E_i \pm \Delta E_i$ are the lattice data.

Another alternative for f^a – not supported by the Appendix – is

$$f^a = (f_3^a + f_4^a \sqrt{S}) \exp(-b_s k'_a S). \quad (3.1)$$

This gives a comparable $\chi^2/\text{d.o.f.}$ of 1.10, but has the inferior feature that f_3^a is not consistent with zero and so in violation of the condition before Eqs. 2.22.

For comparison, Model Ia with its one free parameter k_f in Eq. 2.11 gives from Minuit a $\chi^2/\text{d.o.f.}$ of 3.16 for $k_f = 0.571(12)$. As seen from Table II most of this $\chi^2/\text{d.o.f.}$ comes from the Tetrahedral geometry – especially from the regular Tetrahedra. This was already anticipated at the end of Section II C.

Another comparison is the extreme limit of $f = 1$, where only two-quark potentials are used in Model I. Here all the parameters are fixed, so that this is not a minimization. From Table II it is seen that this gives a very poor representation of the data – especially for E_1 . As expected, the larger the size of the geometry the larger the contribution to $\chi^2/\text{d.o.f.}$, since those are the cases that need most the smaller values of f . For example, with a 6×6 square the model gives $E_{0,1} = -0.124, 0.3721$, whereas from the lattice simulation the corresponding energies are $-0.028(5), 0.065(7)$. This comparison is the basis of our statement in the introduction that multi-quark models containing only the standard two-quark potential of Eq. 2.8 or even the more elaborate form in Eq. 2.24 seem to have no justification for the cases considered here.

B. The continuum limit

The above results are for $\beta = 2.4$, which corresponds to a lattice spacing of $a \approx 0.12$ fm. But, in practice, if the model is to be used, for example, in some type of resonating group approach [4], then it is the continuum limit that is needed. Such an extrapolation would, in principle, require repeating the above lattice calculations with all 6 geometries for a series of increasing values of β . This we do not plan to do here. Instead we exploit the conclusions of Ref. [11]. There it was shown that the four-quark *binding energies* already scale at $\beta = 2.4$, even though the total two- and four-quark energies do not scale separately. Therefore, here we simply assume that, if some binding energy $E(\beta)$ would be scaled to the $\beta = 2.4$ lattice by the transformation $\rho E[\beta, r(\beta)/\rho] \rightarrow E(2.4)$, where $\rho = a(2.4)/a(\beta)$, then we would find that this scaled $E(2.4)$ equals the binding energy $E(r)$ we have calculated directly on the $\beta = 2.4$ lattice. Here the $r(\beta)$ and r are distances in the lattice units of the two different lattices. However, to completely specify the model at some other value of β also needs a knowledge of $v(ij, \beta)$ and $\Delta v(ij, \beta)$ corresponding to the $\beta = 2.4$ expressions in Eqs. 2.24, 2.25. In Ref. [11] it was shown that $v(ij, \beta)$ did not scale. But this is not a problem, since $v(ij, \beta)$ is

documented in the literature for other values of β – see Refs. [14–16]. These can then be scaled in the same way as the $E(\beta)$ to give $v(ij, 2.4)$. However, as found in Ref. [11] we do not expect these $v(ij, 2.4)$ to equal the $v(ij)$ in Eq. 2.24. It remains to specify $\Delta v(ij, \beta)$. Here we simply assume that it scales, so that the values at $\beta = 2.4$ can be used directly in precisely the same way as the binding energies $E(\beta)$. The rationale behind this last assumption is twofold:

- i) In the model it is the *differences* Δv and $v(ij, \beta) - v(kl, \beta)$ that enter as in Eqs. 2.23, 2.27 and 2.28. Therefore, some of the lack of scaling in $v^*(ij, \beta)$ and $v(ij, \beta)$ may well cancel – as in the four-quark binding energies.
- ii) Any lack of scaling that remains will be reflected in a lack of scaling of the a_0, b_0, c_0 multiplying the Δv and V^* 's.

The actual values of β used are 2.5 [14], 2.5 (denoted by 2.5'), 2.635 and 2.74 [15] and 2.85 [16]. In the last case, the authors present two potentials $V(R, 5)$ and $V(\infty)$ that are appropriate for interquark distances R greater than $6a(2.85) \approx 0.17$ fm. Their own preference is $V(R, 5)$ – the one we use.

The results are shown in Table III. The main observation is that, of the free parameters, only k_f appears to be a significant function of β – but even there all values are within the error bars. Therefore, the final version of the model has only four independent parameters.

A similar continuum extrapolation was carried out in Ref. [11]. However there are several major differences compared with the above analysis:

- a) Much less data for each value of β was included in the fits, since only Squares and Tilted Rectangles were analysed. Furthermore, only the E_0 's were fitted. This means that, for each β , there were only 17 pieces of data compared with the present case of 100 pieces covering 6 geometries with both E_0 and E_1 .
- b) All the two- and four-quark energies were simulated at five values of β and interpolated to the same physical sizes. As scaling was found for the binding energies, they are here obtained directly from the $\beta = 2.4$ values.
- c) In order to study scaling and the effect of lattice artefacts, the errors on the data were taken to be purely statistical and so were much smaller than the present ones.
- d) The model analysis was carried out with the 2×2 model. Since this results in a quadratic equation for the energies, the step $E_0 \rightarrow f$ could be done analytically. After this the parameters k_f, k_P, f_0 in Eq. 2.13 were extracted by a fitting procedure.

C. A comparison with other work

A fit with a potential model to some of the above four-quark binding energies was attempted earlier in Ref. [17]. However, there are several important differences compared with the present fit:

- 1) The model contained only 2×2 or 3×3 bases that were essentially constructed from A, B, C .
- 2) The Tetrahedral data was not included.
- 3) Most of the data was for $\beta = 2.4$. Only squares included that from $\beta = 2.5$.
- 4) The overlap factor is that in Eq. 2.13 but with $f_0 = 1$.

For the four geometries S,R,Q,L the area S is the same as that used here in Eq. 2.12 and for geometry NP a curved surface is used. However, the perimeters P are determined by the underlying lattice structure of the geometry and not the usually expected perimeter. For example, for Quadrilaterals, P is the perimeter of the *rectangle* enclosing the quadrilateral. The reason for this choice is an attempt to take into account lattice artefacts, which are present for the smallest configurations. These artefacts are more important in Ref. [17], since configurations containing flux-lines of only one lattice link are included. In the present work the cut-off is at two lattice links. For Squares the outcome in [17] was $k_f = 0.296(11)$ and $k_P = 0.080(2)$, compared with $k_f = 0.571(12)$ using Model Ia above in Subsection II A. The difference between the values of these area constants is because the perimeter term, which is introduced to make artefact corrections for *small* configurations, is still important for large configurations, where rotational symmetry is restored and their presence not needed. For example, with a square of side $d = 5$, the two terms in the exponent of Eq. 2.13 are comparable, even though the lattice data for this square is $-0.042(5)$ and $-0.047(5)$ for the corresponding tilted square. In the lattice simulation the basis states have a similar form compared with the ones used in [17] but with the major difference that they are fuzzed to such an extent that these underlying basis states get transformed. Therefore, in [17], for larger configurations it would be necessary to include many other terms with different perimeters, so that on the average the usual perimeter for a particular geometry would probably

be more appropriate. This latter choice was tested in Ref. [11], where it was found to reduce in the continuum limit to the form of Model Ia. Thus the perimeter term seems to measure lattice artefacts also when the physical perimeter is used.

In Ref. [10], the model in Ref. [17] is extended to include the tetrahedral data by using an 8×8 basis, where now all the states are constructed in terms of the underlying lattice. This, therefore, suffers from the above problem that the results are always dependent on the lattice even for large configurations, where rotational invariance is restored.

IV. CONCLUSIONS

In this article an interquark potential model is constructed than can explain, on the average, the energies of a series of four-quark systems, in which the four static quarks are in any one of six geometries. The input energies used as data for this construction [1] were calculated on a $16^3 \times 32$ lattice for four static SU(2) quarks with $\beta = 2.4$. Two or three basis states were used in the lattice simulation enabling reliable estimates to be made of the lowest two energy eigenvalues ($E_{0,1}$). In all, 100 pieces of data were considered suitable for confronting the model. Other pieces were rejected, if they involved configurations suspected of having strong lattice artefacts (e.g. with states constructed from less than two lattice links) or had vanishingly small binding energies (e.g. elongated rectangles with $r \gg d$).

The full model utilized 6 basis states A, B, C, A^*, B^*, C^* and in its most general form had eight parameters. However, in practice, only 4 of these (k_f, k_a, f_2^a and c_0) need be considered as free when fitting the data.

The parameters that are, perhaps, of most interest are those connected with the ranges of the various interactions, namely, k_f, k_a, k_c . Here we define 'range' as $r_{f,a,c} = \sqrt{1/b_s k_{f,a,c}}$. In Model Ia, where $k_{a,c}$ are effectively infinite, we get $k_f(Ia)=0.57(1)$ i.e. $r_f(Ia) = 5.0$ in lattice units. However, in Model III as the excited states A^*, B^*, C^* are introduced, the interaction between the basic states A, B, C decreases by raising k_f to 1.51 giving $r_f(III) = 3.1$. But at the same time this loss of binding by the direct interaction between A, B, C is compensated by their coupling to the A^*, B^*, C^* states. This coupling is found to have about the *same* range $r_a = 5.1$ as $r_f(Ia)$ above, whereas the direct interaction between the A^*, B^*, C^* states seems, in all fits, to be satisfied with simply a two-quark description without any four-quark correction (i.e. $k_c=0$). The observation that $r_f(Ia) \approx r_a$ suggests that the energy density has a range dictated by the longest range available – namely r_a . Therefore, when the A^*, B^*, C^* states are not explicitly present, as in Model Ia, the only available range $r_f(Ia)$ has to simulate the role of r_a . In the binding energies the contributions from the A^*, B^*, C^* states rapidly dominate over those from the A, B, C states. For example, with squares of side R , the A, B, C states contribute only 85, 40, 10% to the binding energy for $R=2,4,6$ respectively. Of course, at the largest distances (≈ 0.7 fm) the quenched approximation is expected to break down and the role of quark-pair creation to become important. We, therefore, come to the following scenario for the four-quark interaction. At the shortest distances, upto about 0.2 fm, perturbation theory is reasonable with the binding being given mainly by the A, B, C states interacting simply through the two-quark potentials with little effect from four-quark potentials. However, for intermediate ranges, from about 0.2 to 0.5 fm, the four-quark potentials act in such a way as to reduce the effect of the A, B, C states so that the binding is dominated by the A^*, B^*, C^* states, which now interact amongst themselves again simply through the two-quark potentials *with little effect from four-quark potentials*. At larger ranges quark-pair creation can no longer be neglected.

The next step in this work would be to carry out a Resonating Group calculation in the spirit of Ref. [4]. This would then test the universality of the above model by extending it from the 6 geometries discussed here to general four-quark geometries.

V. ACKNOWLEDGEMENT

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Appendix

The justification of Eq. 2.22

In the case of regular tetrahedra a simple model for the $A(13, 24), A^*(13, 24)$ states is as follows:

$$A(13, 24) = N \frac{1}{\sqrt{2}}(ab + cd) \frac{1}{\sqrt{2}}(ef + gh),$$

$$B(14, 23) = N \frac{1}{\sqrt{2}}(aj + ih) \frac{1}{\sqrt{2}}(ek + ld),$$

$$A^*(13, 24) = N^* \frac{1}{\sqrt{2}}(ab - cd) \frac{1}{\sqrt{2}}(ef - gh),$$

$$B^*(14, 23) = N^* \frac{1}{\sqrt{2}}(aj - ih) \frac{1}{\sqrt{2}}(ek - ld), \quad (5.1)$$

where a, b, c, \dots, k, l are links along the sides of the cube defining the regular tetrahedron and N and N^* are normalisation constants. The loops a, b, c, d and e, f, g, h are taken to be in the xz plane and the loops a, j, i, h and e, k, l, d are in the yz plane.

The above definitions guarantee $\langle A(13, 24)A^*(13, 24) \rangle = 0$ etc. and give for the two normalisations:

$$\langle AA \rangle = N^2[1 + 1 + S(xz) + S(xz)]^2/4 \text{ i.e. } N = 1/[1 + S(xz)] \text{ and}$$

$$\langle A^*A^* \rangle = N^{*2}[1 + 1 - S(xz) - S(xz)]^2 \text{ i.e. } N^* = 1/[1 - S(xz)],$$

where $S(xz)$ is a $R \times R$ Square loop of links in the xz plane along the sides of the cube i.e. perimeter is of length $4R$. Other loops needed in this discussion are as follows:

The loop L is two connected faces of the cube in an L -shape i.e. perimeter is of length $6R$,

C is three connected faces of the cube giving a corner i.e. perimeter is also of length $6R$

T is three faces of the cube giving a "table" i.e. perimeter is of length $8R$

With this notation the following overlaps can be written down:

$$\langle A(13, 24)B(14, 23) \rangle = -\frac{1}{2} \frac{1}{4} [2S + 2T + 4C + 8L]/(1 + S)^2 = -f/2,$$

where the initial factor of $-1/2$ is the usual colour factor giving

$$f = \frac{[2S + 2T + 4C + 8L]}{4(1 + S)^2}.$$

Likewise,

$$f^c = \frac{[2S + 2T + 4C - 8L]}{4(1 - S)^2} \quad \text{and} \quad f^a = \frac{[2S + 2T - 4C]}{4(1 - S^2)}.$$

A rough estimate of the relative importance of S, L, C, T can be seen as follows: In a Wilson loop, the further two combinations of links (e.g. $L1, L2$) are separated in Euclidean time, the less is their correlation $\langle L1.L2 \rangle = \exp(-VT)$. Similarly, spatial links are less correlated the further they are apart.

For the Square (S) let side 1 have a factor 1 and the two adjacent sides factors of p and the fourth side p^2 . This makes $S = D.1.p.p^2 = p^4$, where D is some constant and p some correlation factor to adjacent sides with $p < 1$

For the L -shape let the correlation factors have an extra q , when changing direction out of the original side of the cube and an extra r for the final side. In all $L = D.1.p.pq.pqr.pq.p = Dp^5q^3r$ when moving around the perimeter of the L -shape.

For the C -shape the same factor arises as in the L -shape. This suggests $L = C$.

For the T -shape the last set of links is close to original ones, so that the factor on the last link is p^2 as in the S case. In all $T = D.1.p.pq.pqs.p^2.pqs.pq.p = Dp^8q^4s^2$.

For large R we expect $p, q, r, s \ll 1$ which suggests $S \gg (L \text{ or } C) \gg T$.

For small R we expect $p, q, r, s \approx 1$ i.e. $S \approx (L \text{ or } C) \approx T \approx 1$

One way to parametrize these two limits is to take

$S = \exp(-WR^2) \approx (1 - WR^2)$ since it involves a single square of area R^2 .

$L = \exp(-W2R^2) \approx (1 - 2WR^2)$ since it involves two squares of area R^2 .

$T = \exp(-W3R^2) \approx (1 - 3WR^2)$ since it involves 3 such squares.

Keeping only the terms upto $O(R^2)$ in S, L, C, T gives simply $f = 1 - WR^2$, $f^a = f^c = 0$ and is model independent. Whereas the addition of the $O(R^4)$ terms from expanding further the exponentials gives $f^a = WR^2/4$ and $f^c = 1/2$. This last result clearly shows that the $O(R^4)$ terms are not so simple, since we expect $f^c \rightarrow 1$ as $R \rightarrow 0$ – as in the case of f . Similarly these arguments would suggest $k_f = 4f_2^a$. However, the R^2 dependence of f^a is probably correct and so for general geometries we assume $f^a \propto \text{Area}$ as $\text{Area} \rightarrow 0$. For the case of regular tetrahedra, the appropriate area from Eq. 2.12 is $\sqrt{3}R^2$.

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	k_f	k_a	f_2^a	k_c	a_0	b_0	c_0	χ^2/dof
Fix 4	1.51(8)	0.55(3)	0.51(2)	<u>0.0</u>	<u>4.0</u>	<u>1.0</u>	3.9(3)	1.03
Fix 2	1.50(8)	0.57(3)	0.54(7)	0.02(5)	<u>4.0</u>	1.06(32)	4.3(1.3)	1.05
Fix 1	1.61(11)	0.59(5)	0.28(6)	0.00(11)	3.4(8)	0.19(2)	0.78(6)	0.90
Ref. [13]	1.25(6)	0.54(11)	0.66(4)	0.04(20)	4.4(3)	2.2(6)	8.0(4)	1.08

TABLE I. The values of the parameters defining the interaction and fixing $f_1^a = 0$. Those numbers underlined are kept fixed.

Model	Model III		Model Ia		Model I(f=1)	
Geometry	$\chi_A^2(E_0)$	$\chi_A^2(E_1)$	$\chi_A^2(E_0)$	$\chi_A^2(E_1)$	$\chi_A^2(E_0)$	$\chi_A^2(E_1)$
T	0.9	0.7	8.5	7.6	15.8	141.5
S	0.2	1.9	1.2	1.5	102.4	659.6
R	1.1	0.9	1.1	0.5	79.2	527.2
Q	2.7	1.2	1.0	0.3	0.8	6.2
NP	1.0	0.2	1.7	0.2	0.6	16.9
L	0.3	2.6	0.3	2.6	0.3	2.6

TABLE II. The contributions $\chi_A^2(E_{0,1})$ from the two states (E_0, E_1) in each of the six geometries (T, S, R, Q, NP, L) to the total $\chi^2/\text{d.o.f.}$ of 1.03 for Model III, 3.16 for Model Ia and 144 for Model I($f = 1$).

Parameter	$\beta = 2.4$	$\beta = 2.5$	$\beta = 2.5'$	$\beta = 2.635$	$\beta = 2.74$	$\beta = 2.85$
k_f	1.51(8)	1.48(8)	1.50(8)	1.49(8)	1.45(8)	1.43(8)
k_a	0.55(3)	0.56(3)	0.56(3)	0.56(3)	0.57(3)	0.57(3)
f_2^a	0.51(2)	0.51(2)	0.51(2)	0.51(2)	0.51(2)	0.51(2)
c_0	3.9(3)	3.9(3)	3.9(3)	3.9(3)	3.9(3)	3.9(3)
$\chi^2/\text{d.o.f.}$	1.03	1.03	1.11	1.09	1.15	1.07

TABLE III. The values of the parameters defining the interaction fixing $f_1^a = k_c = 0$, $b_0 = 1.0$ and $a_0 = 4.0$ for a series of β 's.